Einstein's Light-Clock Thought Experiment & Derivation of the Time Dilation Formula

The light-clock measures time by the passage of a light beam towards a mirror (tick) and back again (tock). As far as the clock is concerned, it is in an inertial (non-accelerating) reference frame and the light follows its own path back from the mirror (shown offset for clarity). See Diagram 1.

To an observer, the clock appears to be moving in space with velocity v, so that the forward beam and the returning beam form the hypotenuses of two triangles, as in Diagram 2.

Diagram 1 - Clock sees light travel at speed c.

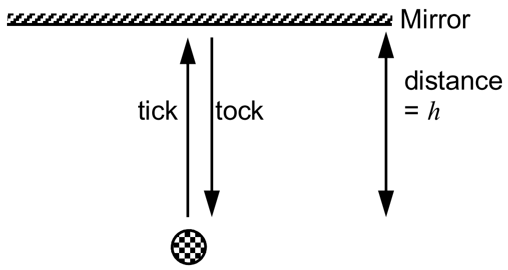
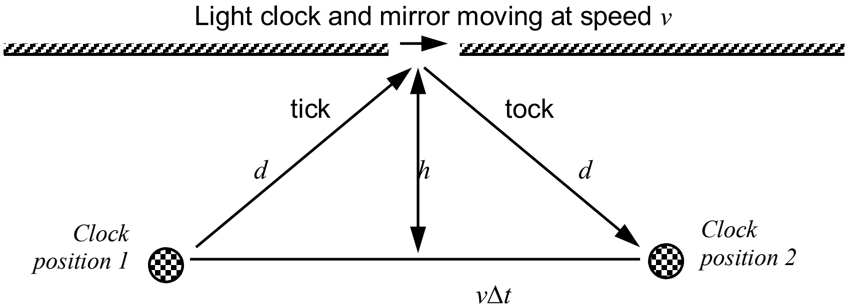


Diagram 2 - Observer sees light travel at speed c.



Light beam travels at speed c, but appears to go a longer distance, $2d$. From the relation that distance = velocity x time, the distance that the whole light clock has moved is $v\Delta t$.
We now have 2 Pythagorean triangles with sides h, d and 1/2 $v\Delta t$.

Basic relations:
velocity = distance / time
distance = velocity x time
time = distance / velocity

The net result of the light clock experiment is that the elapsed time of the moving clock, Δt, from the point of view of a "stationary" observer, is longer than the time that the clock sees, $\Delta t'$.

Let the time interval for the tick-tock from Diagram 1 be called $\Delta t'$. This is also known as "proper time" because it is the interval of time measured in the inertial reference frame of the clock, where the "events" (the tick and tock) are happening.

From Diagram 1:
$$\Delta t' = 2h / c$$
$$h = \Delta t' c / 2 \quad \text{Eq.1}$$

From Diagram 2, the time interval for the tick-tock is $\Delta t = 2d / c$

but, from Pythagoras,
$$d^2 = h^2 + \left(\frac{1}{2} v\Delta t\right)^2$$
$$d = \sqrt{h^2 + \left(\frac{1}{2} v\Delta t\right)^2}$$
$$\Delta t = \frac{2\sqrt{h^2 + \left(\frac{1}{2} v\Delta t\right)^2}}{c} \quad \text{Eq.2}$$

Now, substituting our value for h (Eq.1) into Eq.2, we get:
$$\Delta t = \frac{2\sqrt{(\Delta t' c/2)^2 + (1/2 v\Delta t)^2}}{c}$$

Now simplify to get the relationship between Δt and $\Delta t'$ as follows.

By squaring, expanding, cancelling, rearranging, taking a common factor, dividing and taking a square root ...
$$(\Delta t)^2 = \frac{4\left[(\Delta t' c/2)^2 + (1/2 v\Delta t)^2\right]}{c^2}$$
$$(\Delta t)^2 = \frac{4(\Delta t')^2 c^2 / 4}{c^2} + \frac{4\left(1/4 v^2 (\Delta t)^2\right)}{c^2}$$
$$(\Delta t)^2 = (\Delta t')^2 + v^2 (\Delta t)^2 / c^2$$
$$(\Delta t)^2 - v^2 (\Delta t)^2 / c^2 = (\Delta t')^2$$
$$(\Delta t)^2 (1 - v^2 / c^2) = (\Delta t')^2$$
$$(\Delta t)^2 = \frac{(\Delta t')^2}{(1 - v^2 / c^2)}$$
$$\Delta t = \sqrt{\frac{(\Delta t')^2}{(1 - v^2 / c^2)}}$$
$$\Delta t = \Delta t' / \sqrt{(1 - v^2 / c^2)}$$

... we are able to derive the formula for time dilation for the moving object as perceived by a stationary observer according to special relativity.

The moving clock appears to a stationary observer to be running slow!

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